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Bifurcations of SDOF mechanisms using catastrophe theory

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Abstract

This paper deals with the singularities which occur on the compatibility paths of single degree of freedom finite mechanisms consisting of rigid bars and pin joints. We first examine the analogy between the bifurcations along the compatibility paths of mechanisms and those appeared in the equilibrium paths of elastic structures. Based on it, we propose that the compatibility conditions can be analysed in the same way as the equilibrium equations using the elementary catastrophe theory. A number of mechanism examples are given to illustrate that common cuspoid catastrophe types also exist in the compatibility paths of mechanisms.

Keywords: Catastrophe theory; Catastrophe types; Mechanism with one degree-of-freedom; Compatibility path; Bifurcation

1. Introduction

Some of the modern structural systems are in fact mechanisms. They contain internal mobility so that their shape or geometry can be controlled to suit particular needs. Many large space deployable structures belong to this category because they need to be folded up for transportation and expanded when reaching orbit. A very important part of research into such structures involves design of the internal degree of freedom which allows folding and expansion to take place in a controllable manner. However, some of such designs may end up producing structures with singularities, resulting in that the structure opened up to a configuration which differs from the desirable one. Hence, it is necessary to study singularities in mechanisms.

Singularities are not new to structural engineers. They exist in the equilibrium paths of elastic structural systems. Consider a general conservative structural system, described by the total potential energy function $V(x_i, t_l)$ $(1 \leqslant i \leqslant n;\ 1 \leqslant l \leqslant r)$ where x_i represents a set of n variables, which could be the generalized coordinates, and t_l denotes a set of r control parameters, such as the loading parameter. A set of n equilibrium equations can be obtained when V becomes stationary, i.e.,

$$\partial V/\partial x_i = 0 \quad (i = 1, 2, \ldots, n). \tag{1}$$

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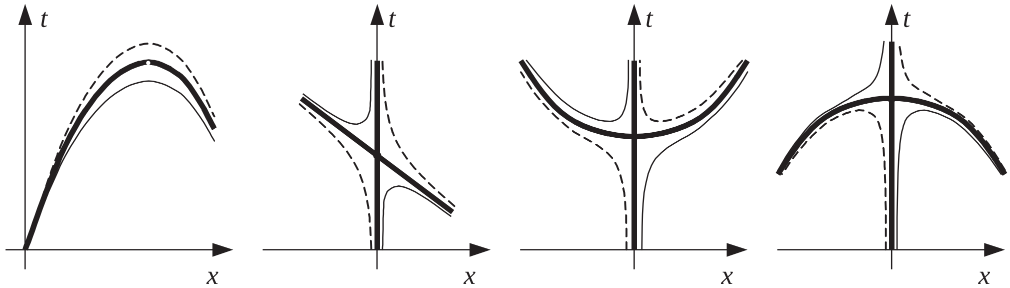

Fig. 1. Four fundamental critical points of equilibrium paths: (a) limit point, (b) asymmetric bifurcation, (c) stable symmetric bifurcation and (d) unstable symmetric bifurcation.

These equations also represent a set of equilibrium paths in the (x_i, t_l) coordinate system. The study of the stability of these paths yields four common modes of instability, and the loss of stability may occur at a limit point or at three modes of bifurcation: namely the asymmetric, symmetric stable, and symmetric unstable, see Fig. 1(a)–(d) (Thompson and Hunt, 1973, 1984).

Similar approach can be applied to study of mechanisms. The motion of a mechanism can be described by its compatibility equations, which define the compatible positions of a mechanism in terms of its state variables, or compatibility paths when the equations are plotted in the state-variable space. Litvin (1980) and Tarnai (1999) discovered that the compatibility paths of some simple four bar chains produced asymmetric bifurcations, just like that in the equilibrium path of an elastic structural system. Tarnai has drawn attention to the striking similarity between the asymmetric bifurcation of equilibrium paths of elastic structures and that of compatibility paths of mechanisms. Lengyel and You (2003) have created an analogy between the two fields and found mechanism examples which produce other bifurcation modes. During the course of this study, two problems were encountered. Firstly, it was found that different set of state variables may not always serve equally. This can be illustrated by a simple planar bar and joint mechanism shown in Fig. 2(a). The mechanism consists of six bars. The length of two supported bars, O_AA and O_BB, is unity while that of the others $\sqrt{2}$. Node D is only allowed to move vertically. So the linkage is symmetric in its basic configuration. If angles α and ϕ are chosen to describe the motion of the linkage, we obtain the symmetric bifurcations shown in Fig. 2(b). However, if ϕ is replaced by x_A, the x coordinate of node A, the graph becomes different, see Fig. 2(c), in which the symmetric bifurcation is no longer visible. Secondly, we discovered that along the straight path in Fig. 2(b), the Jacobian matrix has reduced rank, suggesting the existence of singularity. However, no obvious bifurcation could be observed.

Similar problems appear in the stability theory. Different sets of state variables provide different equilibrium paths, which may even become fundamentally different. For example, in buckling of a cylindrical shell, if the vertical force and the longitudinal compression are chosen as state variables, the planar plot of the compatibility paths is characteristically different from the three-dimensional one obtained by including a third state variable: the radial displacement (von Kármán and Tsien, 1941).

Based on these examples, it seems that the graphs of equilibrium or compatibility paths alone may not be sufficient to reveal the true nature of the behaviour of the object we study. A new approach is required, which, in the stability theory of structures, is provided by the application of the elementary catastrophe theory (Gilmore, 1981; Thompson and Hunt, 1984; Gáspár, 1999).

The catastrophe theory in general was developed by Thom (1975) to study sudden changes in a function due to small changes in the parameters of a system. The elementary catastrophe theory (Gilmore, 1981; Poston and Stewart, 1978) deals with the equilibriums of gradient systems, i.e., systems which could be

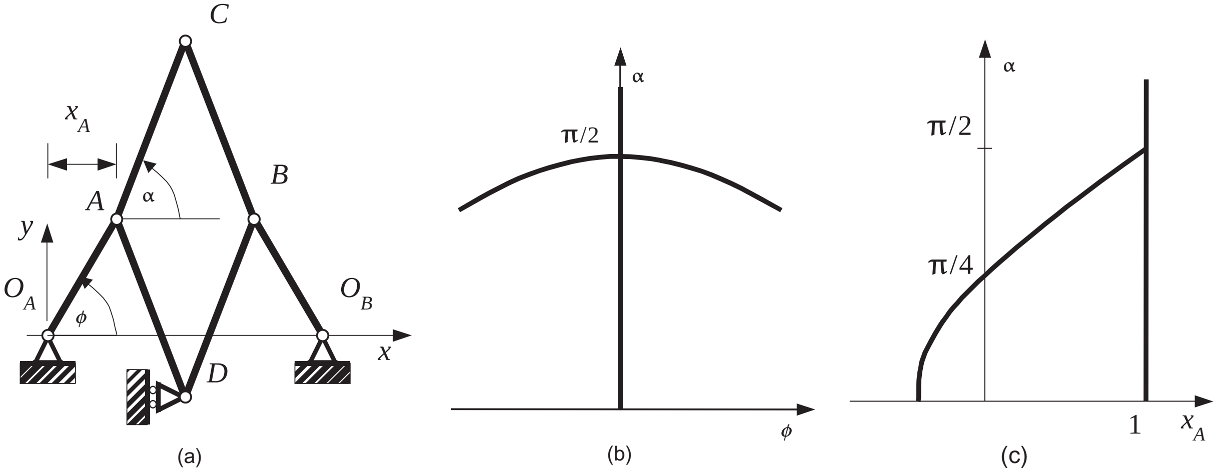

Fig. 2. A six-bar planar mechanism: (a) basic configuration, (b, c) compatibility paths obtained using two different sets of parameters.

described by (1). The stability properties of the equilibrium may be determined from the Hessian matrix of V. If the Hessian matrix is not singular, a suitable diffeomorphism will transform V into the Morse canonical form. If the Hessian is singular, then some of the eigenvalues vanish and the potential can be split to a Morse and a non-Morse part due to Thom splitting lemma. However, it is still possible to find a canonical form for the non-Morse part of the potential. Thom's theorem classifies the typical singularities of forms with less than six parameters. Depending on the number of vanishing eigenvalues, canonical forms of one or two variables are obtained. The one variable forms, called the cuspoid catastrophe types, are listed in Table 1 according to their common names and symbolic notations given by Arnol'd. The equilibrium forms can be easily derived from the canonical forms.

In Table 1, x denotes the single variable and t's are the parameters. The highest order term of the canonical form, independent from the parameters, is called a *catastrophe germ* and the rest are the perturbation terms. If the potential energy function of a structure at a critical point is locally equivalent to one of the canonical forms, then the potential is at a catastrophe point of that type.

The object of this paper is to show that the elementary catastrophe theory can also be used in analysis of the compatibility conditions in mechanisms where singularities occur. The focus of the study is on the bar-and-joint mechanisms which form an important part of in the design of strain-free deployable structures. We will show that all of the cuspoid catastrophe types also exist in the compatibility of mechanisms, and thus, the singularities can be analysed in the same way as in stability of elastic structures.

Table 1
Canonical and equilibrium forms for the cuspoid catastrophe types

Type	Name	Canonical form V	Equilibrium form $\frac{\partial V}{\partial x}$
A_2	Fold	$x^3 + t_1x$	$3x^2 + t_1$
A_3	Cusp	$\pm(x^4 + t_2x^2 + t_1x)$	$\pm(4x^3 + 2t_2x + t_1)$
A_4	Swallowtail	$x^5 + t_3x^3 + t_2x^2 + t_1x$	$5x^4 + 3t_3x^2 + 2t_2x + t_1$
A_5	Butterfly	$\pm(x^6 + t_4x^4 + t_3x^3 + t_2x^2 + t_1x)$	$\pm(6x^5 + 4t_4x^3 + 3t_3x^2 + 2t_2x + t_1)$
A_6	Wigwam	$x^7 + t_5x^5 + t_4x^4 + t_3x^3 + t_2x^2 + t_1x$	$7x^6 + 5t_5x^4 + 4t_4x^3 + 3t_3x^2 + 2t_2x + t_1$

It should be pointed out that the general motions of a rigid body has been a subject of recent research, which were well summarised by Donelan et al. (1999) and Xiang (1995). The study of compatibility presented here is fundamentally different. It gives the behaviour of the system rather than the trajectories of some particular points on a rigid body.

The outline of the paper is as follows. Section 2 explores the similarities of structural systems and mechanisms and proposes a way of how catastrophe theory can be used for the analysis of the compatibility. Section 3 presents mechanisms that correspond to various catastrophe types, such as the *fold*, the *cusp* and the *swallowtail*. Section 4 discusses other catastrophe types. Finally, a summary of the analogy is given in Section 5.

2. Application of catastrophe theory to mechanisms

Here we deal with finite mechanisms with only one degree-of-freedom. Though many variables and constraint equations may be needed to describe the mechanism globally, it is always possible to analyse the local behaviour in terms of only two suitably chosen variables. One constraint condition, i.e., the compatibility equation, is needed that expresses the relationship between the two variables.

In stability of elastic structures, the equilibrium of elastic structures is the gradient of the total potential energy function. Critical points of an equilibrium equation are classified by the local form of the potential energy (Thom's theorem). Similarly, the compatibility condition can be classified on the basis of the analogy between equilibrium and compatibility. Thus, a singular position of a mechanism can be regarded as equivalent to a catastrophe type if the compatibility condition is locally equivalent to the equilibrium forms of that particular catastrophe. For example, for a mechanism with two kinematic state variables and one compatibility condition, if the compatibility condition at a bifurcation point can be transformed to the second equilibrium form in Table 1, then the mechanism is equivalent to *a cusp catastrophe* (A_3).

It should be pointed out that various formulations of potential energy function have been proposed for mechanisms, none yields the compatibility equation as its gradient (Tarnai, 1990; Géradin, 1999). A simple approach would be defining

$$V(x, t_l) = \int F(x, t_l)\,\mathrm{d}x + G(t_l) \tag{2}$$

as the potential function, where F is the compatibility equation and G is a function of parameters only. However, as the compatibility equations usually are highly non-linear, such an integration would yield an extensive function that might have no physical meaning. This reinforces our chosen approach.

In order to express the compatibility condition in the forms listed in Table 1, it needs to be formulated in terms of a variable x and parameters t's. Therefore one of the two state variables should be taken as a variable and the other as a control parameter. Such a distinction can be done by considering the actuation of the mechanism. For instance, if the mechanism is a part of a machinery which is driven by an actuator attached to an element, then the state variable associated with the displacement of that element can be regarded as a control parameter. More parameters are needed when the mechanism has geometric imperfections.

In the following analysis, examples are presented which have different types of catastrophe. We first establish a compatibility function for a given mechanism, then the function is expanded into Taylor series at the critical points. The higher order effects of the parameters are ignored, and therefore the series is truncated accordingly. The remaining items are then compared with the equilibrium forms given in Table 1, as have been done in the structural stability theory.

3. Examples

3.1. The fold catastrophe

Consider the kite-shape four-bar mechanism in Fig. 3(a) that Tarnai (1999) used to demonstrate asymmetric bifurcation. Taking angles α and β as state variables and assuming that the supported bars are rigid, the compatibility condition for the remaining bar is

$$F \equiv \sqrt{(a + b\cos\beta - a\cos\alpha)^2 + (b\sin\beta - a\sin\alpha)^2} - b = 0. \tag{3}$$

Bifurcation occurs at ($\alpha = 0$, $\beta = 0$), see Fig. 3(b). Expanding (3) into Taylor series at that point yields

$$F = \frac{a(a+b)}{2b}\alpha^2 - a\alpha\beta + R_3, \tag{4}$$

where R_3 is the residual term containing terms of third order and higher. Applying a linear transformation:

$$\alpha = u + \frac{2ab}{a^2 + ab + 2b}v, \quad \beta = \frac{a^2 + ab - 2b}{2ab}v. \tag{5}$$

In terms of the new variables the compatibility is written as

$$F = u^2 - \left(\frac{2ab}{a^2 + ab + 2b}\right)^2 v^2 + R_3. \tag{6}$$

Taking u as a variable and choosing a parameter proportional to the second term of (6), the first equilibrium form in Table 1 is obtained. Thus the bifurcation point is equivalent to a *fold catastrophe* (A_2).

Geometric imperfections of the mechanism can also be considered, which lead to new parameters in (3). For instance, assuming that the length of the coupler bar varies by a small ε, the compatibility condition now becomes

$$F \equiv \sqrt{(a + b\cos\beta - a\cos\alpha)^2 + (b\sin\beta - a\sin\alpha)^2} - (b + \varepsilon) = 0. \tag{7}$$

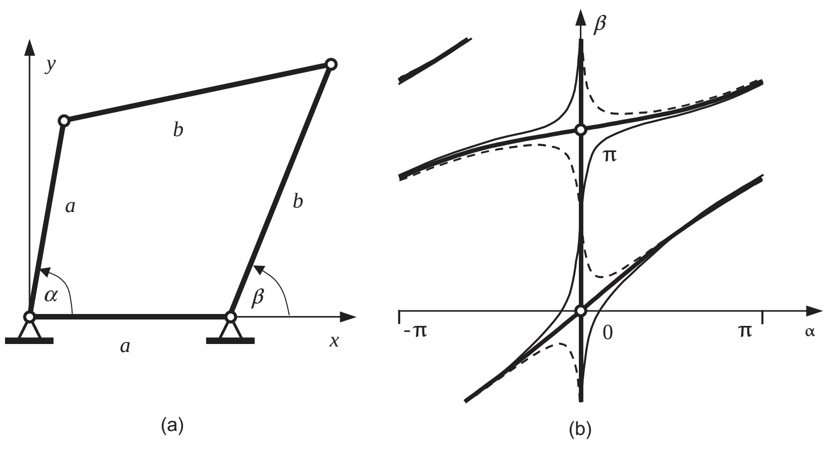

Fig. 3. A kite-shape four-bar mechanism: (a) basic configuration. (b) Compatibility path, thick lines denote the original configuration, continuous and dashed thin lines refer to imperfect geometries obtained by modifying the length of the coupler bar in the negative or positive sense, respectively.

Through the same procedure, we obtain

$$F = u^2 - \left(\frac{2ab}{a^2 + ab + 2b}\right)^2 v^2 + \varepsilon + R_3. \tag{8}$$

Treating u as the variable and the rest as the parameter, the fold catastrophe is obtained again.

Imperfections in other bars can be dealt with as well. However, they give higher order terms, which are omitted if only the linear effect of the imperfections are considered.

The *fold* is a common catastrophe type occurring in stability theory. A typical example is the stability of a rigid-rod spring system, in which the rod in the system is hinged at one end and connected to an inclined spring at the other end (Koiter, 1945).

3.2. The swallowtail catastrophe

Let us revisit the six-bar linkage in Fig. 2(a). Considering its symmetric layout we can set up a single compatibility equation for bar BC using two state variables α and ϕ:

$$F(\alpha,\phi) \equiv \sqrt{(2 - 2\cos\phi - \sqrt{2}\cos\alpha)^2 + (\sqrt{2}\sin\alpha)^2} - \sqrt{2} = 0. \tag{9}$$

It has been shown (Lengyel and You, 2003) that this mechanism produces two types of symmetric bifurcations at ($\phi = 0$, $\alpha = \pi/2$) and ($\phi = 0$, $\alpha = -\pi/2$), see Fig. 4, which occur when nodes A and B

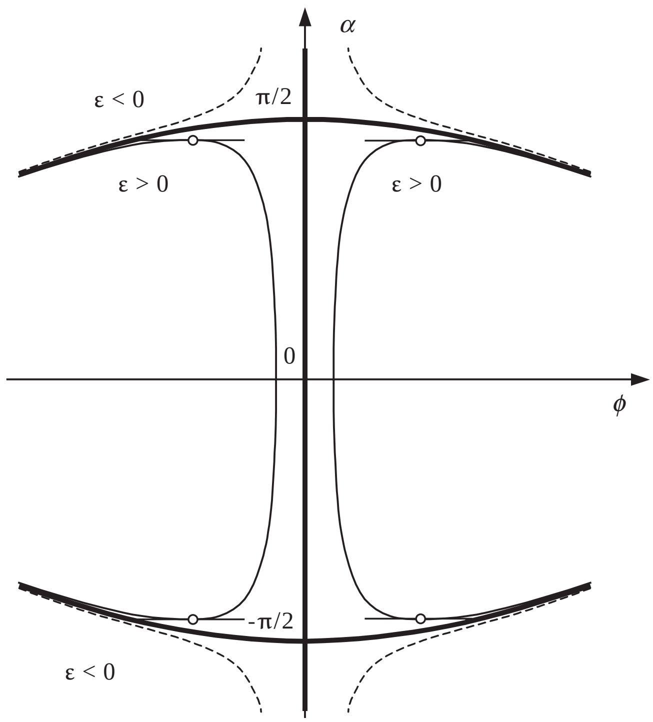

Fig. 4. Compatibility path of the six-bar linkage. Thin and dash lines denote positive and negative imperfections of bar AC, respectively.

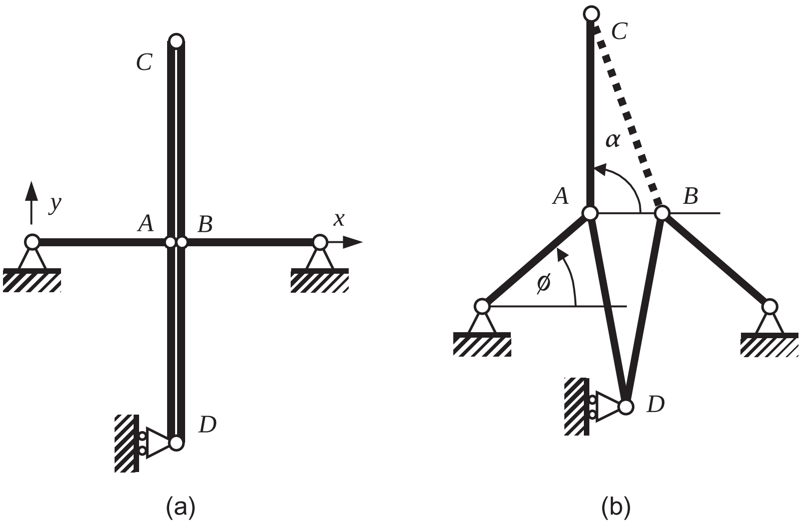

Fig. 5. Configurations of the six-bar linkage corresponding to (a) the bifurcation point ($\alpha = \pi/2$, $\phi = 0$), and (b) near the bifurcation point with an constant α.

coalesce at the origin of the coordinate system and bars AC and BC are pointing either upward or downward along the symmetry line at the centre, as shown in Fig. 5(a).

Expanding (9) into Taylor series at the bifurcation point ($\phi = 0$, $\alpha = \pi/2$) gives

$$F = \frac{\sqrt{2}}{4}\phi^4 + \bar{\alpha}\phi^2 + R_5, \tag{10}$$

where $\bar{\alpha} = \alpha - \pi/2$ and R_5 denotes the terms of order higher than four. Regarding ϕ as a variable, Eq. (12a) becomes equivalent to the third equilibrium form in Table 1, i.e., it is equivalent to a *swallowtail catastrophe* (A_4).

All the other terms in the third equilibrium form of Table 1 can be obtained by introducing geometric imperfections. For simplicity, we shall only introduce ε_1, ε_2 and ε_3 which represent the change in length of bars O_AA, AC and BC, respectively. These imperfections perturb the symmetry of the mechanism so that the compatibility condition now has become a complex and lengthy expression non-linear in its variables and parameters. The Taylor series now is

$$F = \frac{\sqrt{2}}{4}\phi^4 + \left(\bar{\alpha} - \frac{\sqrt{2}}{4}\varepsilon_1\right)\phi^2 + \left(-\frac{1}{4}\varepsilon_1^3\right)\phi + (\varepsilon_2 - \varepsilon_3) + R_5. \tag{11}$$

Let us introduce new parameters t_1, t_2 and t_3:

$$t_1 = \varepsilon_2 - \varepsilon_3, \quad t_2 = -\frac{1}{4}\varepsilon_1^3, \quad t_3 = \bar{\alpha} - \frac{\sqrt{2}}{4}\varepsilon_1. \tag{12a}$$

Eq. (11) now contains all the terms in the third equilibrium form in Table 1.

The order of the compatibility condition of the six bar mechanism is higher than the four bar kite-shaped mechanism. The higher-order term is the result of the symmetry of the perfect configuration at the bifurcation point, see Fig. 5(a). Nodes A and B have the same y coordinate and the distance contains a second-order term of ϕ. When α remains constant, the compatibility condition for BC can be obtained from the right-angled triangle ABC resulting the term ϕ^4, see Fig. 5(b).

The approach based on catastrophe theory presented here helps to analyse and understand the singular behaviour experienced at the straight compatibility path $\phi = 0$, which has been mentioned in Section 1 and shown in Fig. 4. The singularity is demonstrated by a reduced ranked Jacobian matrix (Lengyel, 2003), suggesting that a catastrophe of a certain type may occur along that path. Let us carry out the analysis at a point: ($\phi = 0$, $\alpha = \alpha_0$) where $\alpha_0 \neq \pm\pi/2$ so that the point in consideration is neither of the bifurcation points. The Taylor series obtained will be the same as Eq. (10) except that $\bar{\alpha} = \alpha - \pi/2$, which is now a non-zero constant. Hence the characteristic term of the series is a second-order one, which indicates *a fold catastrophe*. In other words, the points along the straight compatibility path correspond to a catastrophe type which is lower-order than that at the bifurcation points.

The appearance of the fold catastrophe can be easily explained by examining the general form of the swallowtail catastrophe in the three-dimensional parameter space (t_1, t_2, t_3) because lower order singularities occur in the neighbourhood of the catastrophe point (Gilmore, 1981; Gáspár, 1999). In a space formed by t_1, t_2 and t_3 the origin represents the swallowtail catastrophe (A_4); the two-dimensional surfaces are the fold catastrophe (A_2) and the intersection of the surfaces give either a double fold ($A_2 - A_2$) or a cusp catastrophe (A_3).

We can also find other fold catastrophe by suitably chosen imperfections (parameters). For example, when only bar AC is imperfect ($\varepsilon_2 > 0$), Eq. (12a) becomes

$$t_1 = \varepsilon_2, \quad t_2 = 0, \quad t_3 = \bar{\alpha}. \tag{12b}$$

Substituting Eq. (12b) into Eq. (11) gives the fold catastrophe, which are in fact the two limit points shown by the imperfect compatibility paths in Fig. 4.

Producing a cusp catastrophe requires the combined variation of all three parameters. However, it has been found that the solution lacks obvious physical meaning and will be unlikely encountered in practice (Lengyel, 2003). This leads the next example where a different type of mechanism is reported which exhibits the cusp catastrophe.

The swallowtail is one of the less typical catastrophe types in stability theory. Hui and Hansen (1980) has studied structures exhibiting swallowtail catastrophe.

3.3. The cusp catastrophe

The cusp catastrophe requires creation of a third order variation of the compatibility condition in terms of its variable, referring to the second form in Table 1. This can be achieved by considering initially a two bar assembly shown in Fig. 6(a) where A is fixed and both bars AC and BC have unit length. Let node B be connected to a linkage whose motion is second order and third order along axis x and y, respectively, then the bifurcation point still occurs and the required compatibility is also obtained.

A linkage that produces such motion is shown in Fig. 6(b). It has an A-shape and all bars with unit length. The $DEFG$ linkage has one degree-of-freedom and its position is given by angle ϕ, which is the displacement of bar DE from its base position. The motion of the coupler point H in the local coordinate system is

$$\begin{aligned} x_H &= \frac{2\sqrt{3}}{3}\phi^2 + \frac{4}{3}\phi^3 + \frac{11\sqrt{3}}{6}\phi^4 + \cdots, \\ y_H &= \frac{2\sqrt{3}}{3}\phi^3 + 2\phi^4 + \cdots. \end{aligned} \tag{13}$$

A mechanism can be compiled of these two parts as shown in Fig. 6(c). The compatibility condition is written for bar BC and it is easy to show that the Taylor series expansion is equivalent to the equilibrium form of the *cusp catastrophe* in Table 1.

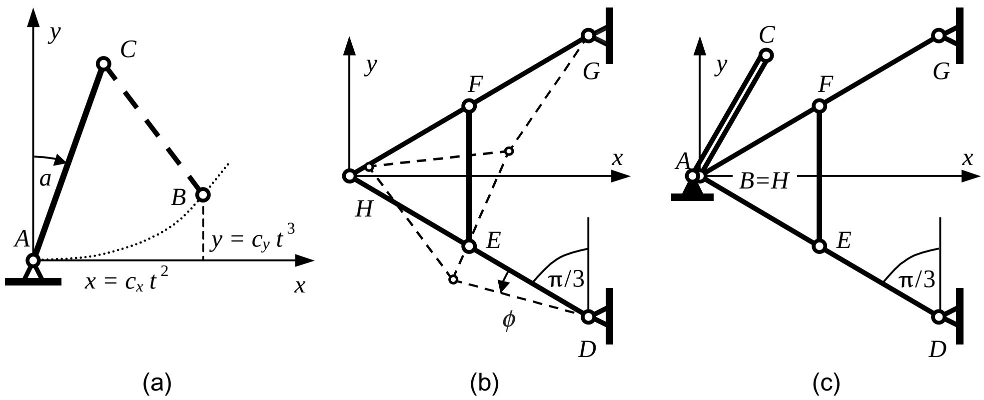

Fig. 6. Construction of linkages with higher-order motion. (a) Two-bar assembly with node B moving on a given trajectory. (b) A-shape mechanism, motion of coupler point H is locally equivalent to that of B. (c) The complete mechanism.

The *cusp* is a common catastrophe type. The hinged axially loaded elastic bar, known as the Euler problem is a simple example in stability theory (Thompson and Hunt, 1984).

4. Other catastrophe types

The generation of various paths is also possible by a general method. Kempe (1876) has proved that it is possible to create a linkwork that traces a planar curve of the nth degree. In order to generate such a curve he applies the multiplication and the addition of angles and the mirroring and the translation of links. Kempe's linkages might prove useful in creating mechanism for higher catastrophe types, however the investigation into this is beyond the scope of this paper.

Another way of producing higher-order catastrophes types arises from degenerate structures. Gáspár (1984, 1999) and Tarnai (2002) have shown structures whose secondary equilibrium paths are horizontal, i.e. neutral in all points. As Gáspár demonstrated, arbitrary cuspoid catastrophe type can be produced by a suitable disturbation of this infinitely degenerate path at the bifurcation point.

A similar phenomenon for mechanisms has been given by Lengyel and You (2003). A particular case of the kite-shape four-bar linkage shown in Fig. 3(a) when all bars have equal length can lead to a graph of the compatibility paths consisting of straight lines given by equations $\alpha = 0$, $\alpha = \beta$ and $\beta = \pi$, instead of the curves shown in Fig. 3(b). In Eq. (7) the parameter β is independent from α and thus arbitrary variation can be possible by a suitable disturbation.

Further study is required to focus on physical imperfections which may produce higher order catastrophe types.

5. Conclusions

Bifurcations of compatibility paths of mechanisms have been studied in the past. The scientific contribution of this paper is that we have analysed the bifurcation points within the compatibility paths of the single degree of freedom (SDOF) mechanisms with the aid of the catastrophe theory. We have shown that bifurcations of several mechanisms indeed correspond to various catastrophe germs. We have been able to demonstrate that the existence of the cuspoid catastrophe types in the compatibility conditions are similar to those of the equilibrium equations in the stability theory for elastic structures. An analogy has been established between the two subjects.

This work has opened the door for detailed study on the kinematic bifurcations of some particular mechanisms and the influence of parameters (imperfections).

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